STABILITY RESULTS FOR SECTIONS OF CONVEX BODIES

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Abstract. It is shown by Makai, Martini, and Ódor that a convex body K, all of whose maximal sections pass through the origin, must be origin-symmetric. We prove a stability version of this result. We also discuss a theorem of Koldobsky and Shane about determination of convex bodies by fractional derivatives of the parallel section function and establish the corresponding stability result.

1. Introduction

Let K be a *convex body* in $\mathbb{R}^n$, i.e. a compact convex set with non-empty interior. More generally, a *body* is a compact subset of $\mathbb{R}^n$ which is equal to the closure of its interior. Throughout the paper, we assume all bodies include the origin as an interior point. Now, we say K is *origin-symmetric* if $K = -K$. The *parallel section function* of K in the direction $\xi \in S^{n-1}$ is defined by

$$A_{K,\xi}(t) = \mathrm{vol}_{n-1}(K \cap \{\xi^\perp + t\xi\}), \quad t \in \mathbb{R}.$$

Here, $\xi^\perp = \{x \in \mathbb{R}^n : \langle x, \xi\rangle = 0\}$ is the hyperplane passing through the origin and orthogonal to the vector ξ.

For the study of central sections it is often more natural to consider a larger class of bodies than the class of convex bodies. Recall that if K is a body containing the origin in its interior and star-shaped with respect to the origin, its *radial function* is defined by

$$\rho_K(\xi) = \max\{a \geq 0 : a\xi \in K\}, \quad \xi \in S^{n-1}.$$

Geometrically, $\rho_K(\xi)$ is the distance from the origin to the point on the boundary in the direction of ξ. If ρ_K is continuous, then K is called a *star body*. Every convex body (with the origin in its interior) is a star body. The *intersection body* of a star body K is the star body IK with radial function

$$\rho_{IK}(\xi) = \mathrm{vol}_{n-1}(K \cap \xi^\perp), \quad \xi \in S^{n-1}.$$

Intersection bodies were introduced by Lutwak in [10] and have been actively studied since then. For example, they played a crucial role in the solution of the Busemann-Petty problem (see [8] for details).

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The *cross-section body* of a convex body K is the star body CK with radial function

$$\rho_{CK}(\xi) = \max_{t \in \mathbb{R}} A_{K,\xi}(t), \quad \xi \in S^{n-1}.$$

Cross-section bodies were introduced by Martini [12]. For properties of these bodies and related questions see [2], [4], [5], [11], [13], [14].

Brunn's theorem asserts that the origin-symmetry of a convex body K implies

$$A_{K,\xi}(0) = \max_{t \in \mathbb{R}} A_{K,\xi}(t)$$

for all $\xi \in S^{n-1}$. In other words, $CK = IK$. The converse statement was proved by Makai, Martini and Ódor [11].

Theorem 1 (Makai, Martini and Ódor)**.** *If K is a convex body in $\mathbb{R}^n$ such that $CK = IK$, then K is origin-symmetric.*

The goal of the present paper is to provide a stability version of Theorem 1. For star bodies K and L in $\mathbb{R}^n$, the *radial metric* is defined as

$$\rho(K, L) = \max_{\xi \in S^{n-1}} |\rho_K(\xi) - \rho_L(\xi)|.$$

We prove the following result.

Theorem 2. *Let K be a convex body in $\mathbb{R}^n$ contained in a ball of radius R, and containing a ball of radius r, where both balls are centred at the origin. If there exists $0 < \varepsilon < \min\left\{ \left(\frac{\sqrt{3}\, r}{6\sqrt{3}\,\pi r + 32\pi} \right)^2, \frac{r^2}{16} \right\}$ so that*

$$\rho(CK, IK) \le \varepsilon,$$

then

$$\rho(K, -K) \le C(n, r, R)\, \varepsilon^q \quad \textit{where} \quad q = \begin{cases} \frac{1}{2} & \textit{if } n = 2, \\ \frac{1}{2(n+1)} & \textit{if } n = 3, 4, \\ \frac{1}{(n-2)(n+1)} & \textit{if } n \ge 5. \end{cases}$$

Here, $C(n, r, R) > 0$ are constants depending on the dimension, r, and R.

Remark. In the proof of Theorem 2, we give the explicit dependency of $C(n, r, R)$ on r and R.

The following corollary is a straightforward consequence of the Lipschitz property of the parallel section function (Lemma 9) and Theorem 2. Roughly speaking: if, for every direction $\xi \in S^{n-1}$, the convex body K has a maximal section perpendicular to ξ that is close to the origin, then K is close to being origin-symmetric.

Corollary 3. *Let K be a convex body in $\mathbb{R}^n$ contained in a ball of radius R, and containing a ball of radius r, where both balls are centred at the origin. Let $L = L(n)$ be the constant given in Lemma 9. If there exists*

$$0 < \varepsilon < \min \left\{ \frac{r}{2}, \frac{3r^3}{LR^{n-1}\left(6\sqrt{3}\pi r + 32\pi\right)^2}, \frac{r^3}{16LR^{n-1}} \right\}$$

so that, for each direction $\xi \in S^{n-1}$, $A_{K,\xi}$ attains its maximum at some $t = t(\xi)$ with $|t(\xi)| \le \varepsilon$, then

$$\rho(K, -K) \le \widetilde{C}(n, r, R)\, \varepsilon^q.$$

Here, $\widetilde{C}(n, r, R) > 0$ are constants depending on the dimension, r, and R, and $q = q(n)$ is the same as in Theorem 2.

The proof of Theorem 2 is given in Section 4 and consists of a sequence of lemmas from Section 3. The main idea is the following. If K is of class C^∞, then we use Brunn's theorem and an integral formula from [3] to show that $\rho(CK, IK)$ being small implies that $\int_{S^{n-1}} \left|A'_{K,\xi}(0)\right|^2 d\xi$ is also small. (Recall that K is called *m-smooth* or C^m, if $\rho_K \in C^m(S^{n-1})$.) If K is not smooth, we approximate it by smooth bodies, for which the above integral is small. Then we use the Fourier transform techniques from [15] and the tools of spherical harmonics similar to those from [6] to finish the proof.

As we will see below, the same methods can be used to obtain a stability version of a result of Koldobsky and Shane [9]. It is well known that the knowledge of $A_{K,\xi}(0)$ for all $\xi \in S^{n-1}$ is not sufficient for determining the body K uniquely, unless K is origin-symmetric. However, Koldobsky and Shane have shown that if $A_{K,\xi}(0)$ is replaced by a fractional derivative of non-integer order of the function $A_{K,\xi}(t)$ at $t = 0$, then this information does determine the body uniquely.

Theorem 4 (Koldobsky and Shane)**.** *Let K and L be convex bodies in $\mathbb{R}^n$. Let $-1 < p < n-1$ be a non-integer, and m be an integer greater than p. If K and L are m-smooth and*

$$A^{(p)}_{K,\xi}(0) = A^{(p)}_{K,\xi}(0),$$

for all $\xi \in S^{n-1}$, then

$$K = L.$$

The following is our stability result.

Theorem 5. *Let K and L be convex bodies in $\mathbb{R}^n$ contained in a ball of radius R, and containing a ball of radius r, where both balls are centred at the origin. Let $-1 < p < n-1$ be a non-integer, and m be an integer greater than p. If K and L are m-smooth and*

$$\sup_{\xi \in S^{n-1}} \left| A^{(p)}_{K,\xi}(0) - A^{(p)}_{L,\xi}(0) \right| \le \varepsilon$$

for some $0 < \varepsilon < 1$, *then*

$$\rho(K,L) \leq C(n,p,r,R)\, \varepsilon^q \quad \textit{where} \quad q = \begin{cases} \frac{2}{n+1} & \textit{if } n \leq 2p+2, \\ \frac{4}{(n-2p)(n+1)} & \textit{if } n > 2p+2. \end{cases}$$

Here, $C(n,p,r,R) > 0$ *are constants depending on the dimension,* p*,* r*, and* R*.*

Remark. In the proof of Theorem 5, we give the explicit dependency of $C(n,p,r,R)$ on r and R. Furthermore, our second result remains true when p is a non-integer greater than $n-1$. However, considering such values for p would make our arguments less clear.

2. Preliminaries

Throughout our paper, the constants

$$\kappa_n := \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} \qquad \text{and} \qquad \omega_n := n \cdot \kappa_n$$

give the volume and surface area of the unit Euclidean ball in $\mathbb{R}^n$, where Γ denotes the Gamma function. Whenever we integrate over Borel subsets of the sphere S^{n-1}, we are using non-normalized *spherical measure*; that is, the $(n-1)$-dimensional Hausdorff measure on $\mathbb{R}^n$, scaled so that the measure of S^{n-1} is ω_n.

Let K be a convex body in $\mathbb{R}^n$ containing the origin in its interior. The *maximal section function* of K is defined by

$$m_K(\xi) = \max_{t\in\mathbb{R}} \mathrm{vol}_{n-1}(K \cap \{\xi^\perp + t\xi\}) = \max_{t\in\mathbb{R}} A_{K,\xi}(t), \qquad \xi \in S^{n-1}.$$

Note that m_K is simply the radial function for the cross-section body CK. For each $\xi \in S^{n-1}$, we let $t_K(\xi) \in \mathbb{R}$ be the closest to zero number such that

$$A_{K,\xi}(t_K(\xi)) = m_K(\xi).$$

Towards the proof of our first stability result, we use the formula

$$\begin{aligned} f_K(t) :&= \frac{1}{\omega_n}\int_{S^{n-1}} A_{K,\xi}(t)\, d\xi \\ &= \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \int_{K\cap\{|x|\geq|t|\}} \frac{1}{|x|}\left(1-\frac{t^2}{|x|^2}\right)^{\frac{n-3}{2}} dx; \end{aligned} \tag{1}$$

refer to Lemma 1.2 in [3] or Lemma 1 in [1] for the proof.

The *Minkowski functional* of K is defined by

$$\|x\|_K = \min\{a \geq 0 : x \in aK\}, \qquad x \in \mathbb{R}^n.$$

It easy to see that $\rho_K(\xi) = \|\xi\|_K^{-1}$ for $\xi \in S^{n-1}$. The latter also allows us to consider ρ_K as a homogeneous degree -1 function on $\mathbb{R}^n \setminus \{0\}$. The *support function* of K is defined by

$$h_K(x) = \sup_{y\in K}\langle x, y\rangle, \qquad x \in \mathbb{R}^n.$$

The function h_K is the Minkowski functional for the polar body K° associated with K. Given another convex body L in $\mathbb{R}^n$, define

$$\delta_2(K,L) = \left(\int_{S^{n-1}} |h_K(\xi) - h_L(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}$$

and

$$\delta_\infty(K,L) = \sup_{\xi \in S^{n-1}} |h_K(\xi) - h_L(\xi)| \, .$$

These functions are, respectively, the L^2 and *Hausdorff metrics* for convex bodies in $\mathbb{R}^n$. The following theorem, due to Vitale [17], relates these metrics; refer to Proposition 2.3.1 in [7] for the proof.

Theorem 6. *Let K and L be convex bodies in $\mathbb{R}^n$, and let D denote the diameter of $K \cup L$. Then*

$$\frac{2\kappa_{n-1} D^{1-n}}{n(n+1)} \, \delta_\infty(K,\, L)^{n+1} \leq \delta_2(K,\, L)^2 \leq \omega_n \, \delta_\infty(K,L)^2.$$

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be any n-tuple of non-negative integers. We will use the notation

$$[\alpha] := \sum_{j=1}^{n} \alpha_j$$

to define the differential operator

$$\frac{\partial^{[\alpha]}}{\partial x^\alpha} := \frac{\partial^{[\alpha]}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

We let $\mathcal{S}(\mathbb{R}^n)$ denote the space of Schwartz test functions; that is, functions in $C^\infty(\mathbb{R}^n)$ for which all derivatives decay faster than any rational function. The Fourier transform of $\phi \in \mathcal{S}(\mathbb{R}^n)$ is a test function $\mathscr{F}\phi$ defined by

$$\mathscr{F}\phi(x) = \widehat{\phi}(x) = \int_{\mathbb{R}^n} \phi(y) e^{-i\langle x,y \rangle} \, dy, \qquad x \in \mathbb{R}^n.$$

The continuous dual of $\mathcal{S}(\mathbb{R}^n)$ is denoted as $\mathcal{S}'(\mathbb{R}^n)$, and elements of $\mathcal{S}'(\mathbb{R}^n)$ are referred to as distributions. The action of $f \in \mathcal{S}'(\mathbb{R}^n)$ on a test function ϕ is denoted as $\langle f, \phi \rangle$. The Fourier transform of f is a distribution $\widehat{f}$ defined by

$$\langle \widehat{f}, \phi \rangle = \langle f, \widehat{\phi} \rangle, \qquad \phi \in \mathcal{S}(\mathbb{R}^n);$$

$\widehat{f}$ is well-defined as a distribution because $\mathscr{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a continuous and linear bijection.

For any $f \in C(S^{n-1})$ and $p \in \mathbb{C}$, the $-n+p$ homogeneous extension of f is given by

$$f_p(x) = |x|^{-n+p} \, f\left(\frac{x}{|x|} \right), \qquad x \in \mathbb{R}^n \backslash \{0\}.$$

When $\mathcal{R}p > 0$, f_p is locally integrable on $\mathbb{R}^n$ with at most polynomial growth at infinity. In this case, f_p is a distribution on $\mathcal{S}(\mathbb{R}^n)$ acting by integration, and we may consider its Fourier transform. Goodey, Yaskin, and Yaskina show in [6] that, for $f \in C^\infty(S^{n-1})$, the additional restriction $\mathcal{R}p < n$ ensures the action of $\widehat{f_p}$ is also by integration, with $\widehat{f_p} \in C^\infty(\mathbb{R}^n \backslash \{0\})$.

We make extensive use of the mapping $I_p : C^\infty(S^{n-1}) \to C^\infty(S^{n-1})$ defined in [6], which sends a function f to the restriction of $\widehat{f_p}$ to S^{n-1}. For $0 < \mathcal{R}p < n$ and $m \in \mathbb{Z}^{\geq 0}$, Goodey, Yaskin and Yaskina show I_p has an eigenvalue $\lambda_m(n,p)$ whose eigenspace includes all spherical harmonics of degree m and dimension n. These eigenvalues are given explicitly in the following lemma; refer to [6] for the proof.

Lemma 7. *If $0 < \mathcal{R}p < n$, then the eigenvalues $\lambda_m(n,p)$ are given by*

$$\lambda_m(n,p) = \frac{2^p \pi^{\frac{n}{2}} (-1)^{\frac{m}{2}} \Gamma\left(\frac{m+p}{2}\right)}{\Gamma\left(\frac{m+n-p}{2}\right)} \quad \textit{if } m \textit{ is even,}$$

and

$$\lambda_m(n,p) = i\,\frac{2^p \pi^{\frac{n}{2}} (-1)^{\frac{m-1}{2}} \Gamma\left(\frac{m+p}{2}\right)}{\Gamma\left(\frac{m+n-p}{2}\right)} \quad \textit{if } m \textit{ is odd.}$$

The *spherical gradient* of $f \in C(S^{n-1})$ is the restriction of $\nabla f\left({}^{x}/_{|x|}\right)$ to S^{n-1}. It is denoted by $\nabla_o f$.

An extensive discussion on spherical harmonics is given in [7]. A *spherical harmonic* Q of dimension n is a harmonic and homogeneous polynomial in n variables whose domain is restricted to S^{n-1}. We say Q is of degree m if the corresponding polynomial has degree m. The collection $\mathcal{H}_m^n$ of all spherical harmonics with dimension n and degree m is a finite dimensional Hilbert space with respect to the inner product for $L^2(S^{n-1})$. If, for each $m \in \mathbb{Z}^{\geq 0}$, $\mathcal{B}_m$ is an orthonormal basis for $\mathcal{H}_m^n$, then the union of all $\mathcal{B}_m$ is an orthonormal basis for $L^2(S^{n-1})$. Given $f \in L^2(S^{n-1})$, and defining

$$\sum_{Q \in \mathcal{B}_m} \langle f, Q \rangle \, Q =: Q_m \in \mathcal{H}_m^n,$$

we call $\sum_{m=0}^\infty Q_m$ the *condensed harmonic expansion* for f. The condensed harmonic expansion does not depend on the particular orthonormal bases chosen for each $\mathcal{H}_m^n$.

Let $m \in \mathbb{N} \cup \{0\}$, and let $h : \mathbb{R} \to \mathbb{C}$ be an integrable function which is m-smooth in a neighbourhood of the origin. For $p \in \mathbb{C} \backslash \mathbb{Z}$ such that $-1 < \mathcal{R}p < m$, we define the *fractional derivative* of the order p of h at zero

as

$$h^{(p)}(0) = \frac{1}{\Gamma(-p)} \int_0^1 t^{-1-p} \left(h(-t) - \sum_{k=0}^{m-1} \frac{(-1)^k h^{(k)}(0)}{k!} t^k \right) dt$$
$$+ \frac{1}{\Gamma(-p)} \int_1^\infty t^{-1-p} h(-t)\, dt + \frac{1}{\Gamma(-p)} \sum_{k=0}^{m-1} \frac{(-1)^k h^{(k)}(0)}{k!(k-p)}.$$

Given the simple poles of the Gamma function, the fractional derivatives of h at zero may be analytically extended to the integer values $0, \dots, m-1$, and they will agree with the classical derivatives.

Let K be an infinitely smooth convex body. By Lemma 2.4 in [8], $A_{K,\xi}$ is infinitely smooth in a neighbourhood of $t = 0$ which is uniform with respect to $\xi \in S^{n-1}$. With the exception of a sign difference, the equality

$$A^{(p)}_{K,\xi}(0) = \frac{\cos\left(\frac{p\pi}{2}\right)}{2\pi(n-1-p)} \left(\|x\|_K^{-n+1+p} + \|-x\|_K^{-n+1+p} \right)^\wedge (\xi) \tag{2}$$
$$+ i\, \frac{\sin\left(\frac{p\pi}{2}\right)}{2\pi(n-1-p)} \left(\|x\|_K^{-n+1+p} - \|-x\|_K^{-n+1+p} \right)^\wedge (\xi),$$

was proven by Ryabogin and Yaskin in [15] for all $\xi \in S^{n-1}$ and $p \in \mathbb{C}$ such that $-1 < Re(p) < n-1$. The sign difference results from their use of $h(x)$ rather than $h(-x)$ in the definition of fractional derivatives.

3. Auxiliary Results

We first prove some auxiliary lemmas.

Lemma 8. *Let m be a non-negative integer. Let K be an m-smooth convex body in $\mathbb{R}^n$ contained in a ball of radius R, and containing a ball of radius r, where both balls are centred at the origin. There exists a family $\{K_\delta\}_{0<\delta<1}$ of infinitely smooth convex bodies in $\mathbb{R}^n$ which approximate K in the radial metric as δ approaches zero, with*

$$B_{(1+\delta)^{-1}r}(0) \subset K_\delta \subset B_{(1-\delta)^{-1}R}(0).$$

Furthermore,

$$\lim_{\delta\to 0} \sup_{\xi\in S^{n-1}} \sup_{|t|\le \frac{r}{4}} \left| A_{K,\xi}(t) - A_{K_\delta,\xi}(t) \right| = 0,$$

and

$$\lim_{\delta\to 0} \sup_{\xi\in S^{n-1}} \left| A^{(p)}_{K_\delta,\xi}(0) - A^{(p)}_{K,\xi}(0) \right| = 0$$

for every $p \in \mathbb{R}$, $-1 < p \le m$.

Proof. For each $0 < \delta < 1$, let $\phi_\delta : [0,\infty) \to [0,\infty)$ be a C^∞ function with support contained in $[\delta/2, \delta]$, and

$$\int_{\mathbb{R}^n} \phi_\delta\big(|z|\big)\, dz = 1.$$

It follows from Theorem 3.3.1 in [16] that there is a family $\{K_\delta\}_{0<\delta<1}$ of C^∞ convex bodies in $\mathbb{R}^n$ such that

$$\|x\|_{K_\delta} = \int_{\mathbb{R}^n} \big\|x + |x|z\big\|_K \phi_\delta\big(|z|\big)\, dz,$$

and

$$\lim_{\delta\to 0} \sup_{\xi\in S^{n-1}} \big|\|\xi\|_{K_\delta} - \|\xi\|_K\big| = 0.$$

For each $\xi \in S^{n-1}$ and $z \in \mathbb{R}^n$ with $|z| \le \delta$, we have

$$\big\|\xi + |\xi|z\big\|_K = \|\xi + z\|_K = \|\lambda\eta\|_K = \lambda\|\eta\|_K$$

for some $\eta \in S^{n-1}$ and $0 < 1-\delta \le \lambda \le 1+\delta$. It then follows from the support of ϕ_δ and the inequality $R^{-1} \le \|\eta\|_K \le r^{-1}$ that

$$\|\xi\|_{K_\delta} = \int_{\mathbb{R}^n} \|\xi + z\|_K \phi_\delta\big(|z|\big)\, dz \le (1+\delta)r^{-1}$$

and

$$\|\xi\|_{K_\delta} = \int_{\mathbb{R}^n} \|\xi + z\|_K \phi_\delta\big(|z|\big)\, dz \ge (1-\delta)R^{-1},$$

which gives

$$B_{(1+\delta)^{-1}r}(0) \subset K_\delta \subset B_{(1-\delta)^{-1}R}(0).$$

This containment, with the limit of the difference of Minkowski functionals above, implies

$$\lim_{\delta\to 0} \sup_{\xi\in S^{n-1}} \big|\rho_{K_\delta}(\xi) - \rho_K(\xi)\big| = 0. \tag{3}$$

Therefore, $\{K_\delta\}_{0<\delta<1}$ approximate K with respect to the radial metric.

Furthermore, the radial functions $\{\rho_{K_\delta}\}_{0<\delta<1}$ approximate ρ_K in $C^m(S^{n-1})$. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be any n-tuple of non-negative integers such that $1 \le [\alpha] \le m$, and consider the function

$$f(y,z) := \frac{\partial^{[\alpha]}}{\partial x^\alpha}\big\|x + |x|z\big\|_K\Big|_{x=y}.$$

Observe that f is uniformly continuous on

$$\big\{y \in \mathbb{R}^n, 2^{-1} \le |y| \le 2\big\} \times \big\{z \in \mathbb{R}^n, |z| \le 2^{-1}\big\}$$

since K is m-smooth. Therefore, we have

$$\frac{\partial^{[\alpha]}}{\partial x^\alpha}\big(\|x\|_{K_\delta} - \|x\|_K\big)\Big|_{x=\xi} = \int_{\mathbb{R}^n} \frac{\partial^{[\alpha]}}{\partial x^\alpha}\Big(\big\|x + |x|z\big\|_K - \|x\|_K\Big)\Big|_{x=\xi} \phi_\delta\big(|z|\big)\, dz$$

for all $\xi \in S^{n-1}$ and $\delta < 1/2$, which implies

$$\sup_{\xi\in S^{n-1}} \left|\frac{\partial^{[\alpha]}}{\partial x^\alpha}\big(\|x\|_{K_\delta} - \|x\|_K\big)\Big|_{x=\xi}\right| \le \sup_{\xi\in S^{n-1}} \sup_{|z|<\delta} \big|f(\xi,z) - f(\xi,0)\big|.$$

Noting that $\left|(\xi,z)-(\xi,0)\right| = |z| < \delta$, the uniform continuity of f then implies

$$\lim_{\delta\to 0}\sup_{\xi\in S^{n-1}}\left|\frac{\partial^{[\alpha]}}{\partial x^{\alpha}}\big(\|x\|_{K_\delta}-\|x\|_K\big)\Big|_{x=\xi}\right| = 0. \tag{4}$$

It follows from the relation $\rho_K(x)=\|x\|_K^{-1}$ that $\frac{\partial^{[\alpha]}}{\partial x^{\alpha}}\rho_K\big|_{x=\xi}$ may be expressed as a finite linear combination of terms of the form

$$\rho_K^{d+1}(\xi)\prod_{j=0}^{d}\frac{\partial^{[\beta_j]}}{\partial x^{\beta_j}}\|x\|_K\Big|_{x=\xi},$$

where $d\in\mathbb{Z}^{\geq 0}$, and each β_j is an n-tuple of non-negative integers such that $[\beta_j]\geq 1$ and $[\alpha]=\sum_{j=0}^{d}[\beta_j]$. Of course, $\frac{\partial^{[\alpha]}}{\partial x^{\alpha}}\rho_{K_\delta}\big|_{x=\xi}$ may be expressed similarly. Equations (3) and (4) then imply

$$\lim_{\delta\to 0}\sup_{\xi\in S^{n-1}}\left|\frac{\partial^{[\alpha]}}{\partial x^{\alpha}}\big(\rho_{K_\delta}-\rho_K\big)\Big|_{x=\xi}\right| = 0, \tag{5}$$

once we note that ρ_K and the partial derivatives of $\|x\|_K$, up to order m, are bounded on S^{n-1}.

Our next step is to uniformly approximate the parallel section function $A_{K,\xi}$. Fix $\xi\in S^{n-1}$, and define the hyperplane

$$H_t=\xi^{\perp}+t\xi$$

for any $t\in\mathbb{R}$ such that $|t|<r$. Let S^{n-2} denote the Euclidean sphere in H_t centred at $t\xi$, and let $\rho_{K\cap H_t}$ denote the radial function for $K\cap H_t$ with respect to $t\xi$ on S^{n-2}. Then, for $|t|<r$,

$$A_{K,\xi}(t)=\frac{1}{n-1}\int_{S^{n-2}}\rho_{K\cap H_t}^{n-1}(\theta)\,d\theta. \tag{6}$$

For $|t|<r/2$ and $0<\delta<1$, $A_{K_\delta,\xi}(t)$ may be expressed similarly. Fixing $\theta\in S^{n-2}$, and with angles α and β as in Figure 1, we have

$$\left|\rho_{K\cap H_t}(\theta)-\rho_{K_\delta\cap H_t}(\theta)\right|\leq\frac{\sin\beta}{\sin\alpha}\left|\rho_K(\eta_1)-\rho_{K_\delta}(\eta_1)\right|.$$

By restricting to $|t|\leq r/4$, α may be bounded away from zero and π. Indeed, if $\alpha<\pi/2$, then

$$\tan\alpha\geq\frac{r/2-|t|}{R}\geq\frac{r}{4R},$$

and if $\alpha>\pi/2$, then

$$\tan(\pi-\alpha)\geq\frac{r/2+|t|}{R}\geq\frac{r}{2R}.$$

Therefore

$$0<\arctan\left(\frac{r}{4R}\right)\leq\alpha\leq\pi-\arctan\left(\frac{r}{4R}\right)<\pi.$$

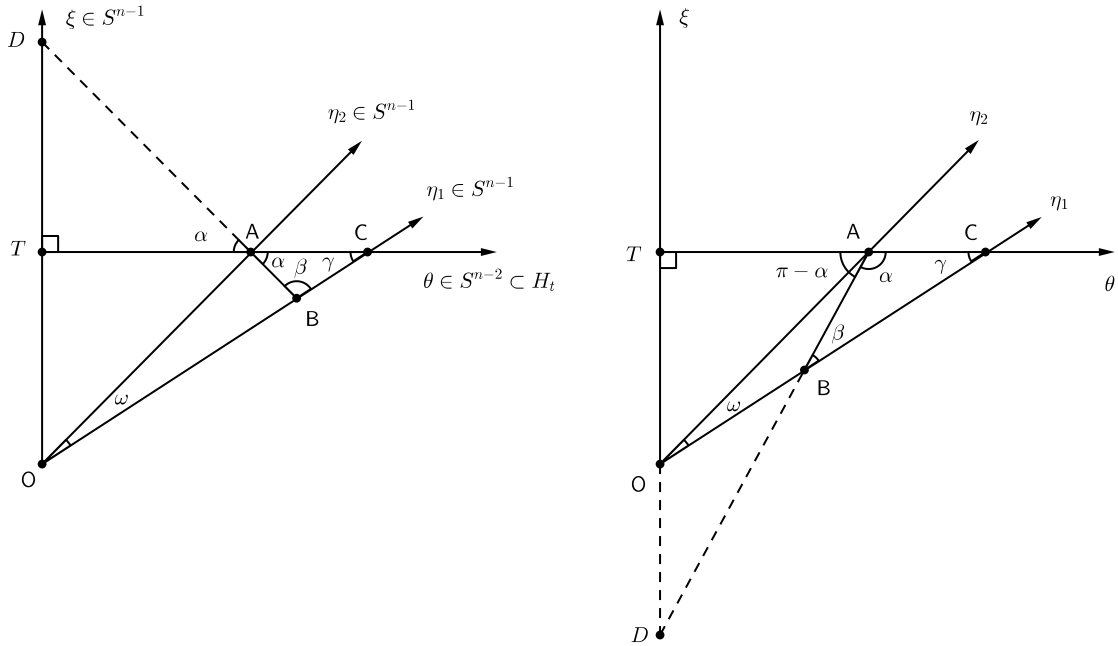

FIGURE 1. The diagrams represent two extremes: when the angle α is small ($\alpha < \pi/2$), and when it is large ($\alpha > \pi/2$). The point O represents the origin in $\mathbb{R}^n$, and $\left|\overline{OT}\right| = t$ where $0 \leq t \leq r/4$. The points A and C are the boundary points for K and K_δ in the direction θ, with two obvious possibilities: either $\left|\overline{TA}\right| = \rho_{K \cap H_t}(\theta)$ and $\left|\overline{TC}\right| = \rho_{K_\delta \cap H_t}(\theta)$, or the opposite. The point B is a boundary point for the same convex body as A, but in the direction η_1. The point D lies outside of the convex body for which A and B are boundary points.

We now have

$$\left|\rho_{K\cap H_t}(\theta) - \rho_{K_\delta \cap H_t}(\theta)\right| \leq \frac{1}{\sin\left(\arctan\left(\frac{r}{4R}\right)\right)} \sup_{\eta \in S^{n-1}} \left|\rho_K(\eta) - \rho_{K_\delta}(\eta)\right|, \quad (7)$$

where the upper bound is independent of $\xi \in S^{n-1}$, t with $|t| \leq r/4$, and $\theta \in S^{n-2}$. This inequality, the integral expression (6), and equation (3) imply

$$\lim_{\delta \to 0} \sup_{\xi \in S^{n-1}} \sup_{|t| \leq \frac{r}{4}} \left|A_{K,\xi}(t) - A_{K_\delta,\xi}(t)\right| = 0.$$

Lemma (2.4) in [8] establishes the existence of a small neighbourhood of $t = 0$, independent of $\xi \in S^{n-1}$, on which $A_{K,\xi}$ is m-smooth. The following is an elaboration of Koldobsky's proof, so that we may uniformly approximate the derivatives of $A_{K,\xi}$. Again fix $\xi \in S^{n-1}$, and fix $\theta \in S^{n-2} \subset H_t$. Let

$\rho_{K,\theta}$ denote the m-smooth restriction of ρ_K to the two dimensional plane spanned by ξ and θ, and consider $\rho_{K,\theta}$ as a function on $[0,2\pi]$, where the angle is measured from the positive θ-axis. A right triangle then gives the equation

$$\rho^2_{K\cap H_t}(\theta)+t^2=\rho^2_{K,\theta}\left(\arctan\left(\frac{t}{\rho_{K\cap H_t}(\theta)}\right)\right),$$

which we can use to implicitly differentiate $y(t):=\rho_{K\cap H_t}(\theta)$ as a function of t. Indeed,

$$F(t,y):=y^2+t^2-\rho^2_{K,\theta}\left(\arctan\left(\frac{t}{y}\right)\right)$$

is differentiable away from $y=0$, with

$$F_y(t,y)=2y+\frac{2t}{y^2+t^2}\rho_{K,\theta}\left(\arctan\left(\frac{t}{y}\right)\right)\rho'_{K,\theta}\left(\arctan\left(\frac{t}{y}\right)\right).$$

The containment $B_r(0)\subset K\subset B_R(0)$ implies $\rho_{K,\theta}$ is bounded above on S^{n-1} by R, and

$$\rho_{K\cap H_t}(\theta)\geq\frac{\sqrt{15}\,r}{4}$$

for $|t|\leq r/4$. If

$$M=1+\sup_{\xi\in S^{n-1}}\left|\nabla_o\rho_K(\xi)\right|<\infty,$$

and $\lambda\in\mathbb{R}$ is a constant such that

$$0<\lambda<\min\left\{\frac{15\sqrt{15}\,r^3}{128RM},\frac{r}{4}\right\},$$

then

$$\left|F_y\big(t,\rho_{K\cap H_t}(\theta)\big)\right|>\frac{\sqrt{15}\,r}{4}$$

for $|t|\leq\lambda$. Therefore, by the Implicit Function Theorem, $y(t)=\rho_{K\cap H_t}(\theta)$ is differentiable on $(-\lambda,\lambda)$, with

$$y'(t)=\frac{\rho_{K,\theta}\left(\arctan\left(\frac{t}{y}\right)\right)\rho'_{K,\theta}\left(\arctan\left(\frac{t}{y}\right)\right)\left(y^2+t^2\right)^{-1}y-t}{y+t\rho_{K,\theta}\left(\arctan\left(\frac{t}{y}\right)\right)\rho'_{K,\theta}\left(\arctan\left(\frac{t}{y}\right)\right)\left(y^2+t^2\right)^{-1}}.$$

Recursion shows that $\rho_{K\cap H_t}(\theta)$ is m-smooth on $(-\lambda,\lambda)$, independent of $\xi\in S^{n-1}$ and $\theta\in S^{n-2}$. It follows from the integral expression (6) that $A_{K,\xi}$ is m-smooth on $(-\lambda,\lambda)$ for every $\xi\in S^{n-1}$. This argument also shows that $A_{K_\delta,\xi}$ is m-smooth on the same interval, for $\delta>0$ small enough. Using the resulting expressions for the derivatives of $A_{K,\xi}$ and $A_{K_\delta,\xi}$, and applying equations (3), (5), and the inequality (7), we have

$$\lim_{\delta\to 0}\sup_{\xi\in S^{n-1}}\sup_{|t|\leq\lambda}\left|A^{(k)}_{K,\xi}(t)-A^{(k)}_{K_\delta,\xi}(t)\right|=0$$

for $k = 1, \dots, m$.

Finally, for any $p \in \mathbb{R}$ such that $-1 < p < m$ and $p \neq 0, 1, \dots, m-1$, we will uniformly approximate $A_{K,\xi}^{(p)}(0)$. With $\lambda > 0$ as chosen above, we have

$$\begin{aligned} A_{K,\xi}^{(p)}(0) = &\frac{1}{\Gamma(-p)} \int_0^\lambda t^{-1-p} \left(A_{K,\xi}(-t) - \sum_{k=0}^{m-1} \frac{(-1)^k A_{K,\xi}^{(k)}(0)}{k!} t^k \right) dt \\ &+ \frac{1}{\Gamma(-p)} \int_\lambda^\infty t^{-1-p} A_{K,\xi}(-t)\, dt + \frac{1}{\Gamma(-p)} \sum_{k=0}^{m-1} \frac{(-1)^k \lambda^{k-p} A_{K,\xi}^{(k)}(0)}{k!(k-p)}. \end{aligned}$$

The first integral in this equation can be rewritten as

$$\int_0^\lambda t^{-1-p} \int_0^t \frac{A_{K,\xi}^{(m)}(-z)}{(m-1)!} (t-z)^{m-1}\, dz\, dt,$$

using the integral form of the remainder in Taylor's Theorem. We also have

$$\begin{aligned} &\int_\lambda^\infty t^{-1-p} A_{K,\xi}(-t)\, dt \\ &= \int_{K \cap \{\langle x, -\xi \rangle \geq \lambda\}} \langle x, -\xi \rangle^{-1-p}\, dx \\ &= \int_{B_K(\xi)} \langle \eta, -\xi \rangle^{-1-p} \int_{\lambda \langle \eta, -\xi \rangle^{-1}}^{\rho_K(\eta)} r^{n-2-p}\, dr\, d\eta \\ &= \frac{1}{n-1-p} \int_{B_K(\xi)} \left(\langle \eta, -\xi \rangle^{-1-p} \rho_K^{n-1-p}(\eta) - \lambda^{n-1-p} \langle \eta, -\xi \rangle^{-n} \right) d\eta, \end{aligned}$$

where

$$B_K(\xi) = \left\{ \eta \in S^{n-1} \middle| \langle \eta, \xi \rangle < 0 \text{ and } \rho_K(\eta) \geq \lambda \langle \eta, -\xi \rangle^{-1} \right\}.$$

Therefore, with the set $B_{K_\delta}(\xi)$ defined similarly, we have

$$\begin{aligned}
&\left|A^{(p)}_{K,\xi}(0) - A^{(p)}_{K_\delta,\xi}(0)\right| \cdot \left|\Gamma(-p)\right| \\
&\le \frac{1}{(m-1)!}\left(\sup_{|z|\le\lambda}\left|A^{(m)}_{K,\xi}(z) - A^{(m)}_{K_\delta,\xi}(z)\right|\right)\int_0^\lambda\int_0^t t^{-1-p}(t-z)^{m-1}\,dz\,dt \quad (8)\\
&\quad + \left(\sup_{\eta\in S^{n-1}}\left|\rho_K^{n-1-p}(\eta) - \rho_{K_\delta}^{n-1-p}(\eta)\right|\right)\int_{B_K(\xi)\cap B_{K_\delta}(\xi)}\frac{\langle\eta,-\xi\rangle^{-1-p}}{|n-1-p|}\,d\eta \quad (9)\\
&\quad + \int_{B_K(\xi)\setminus B_{K_\delta}(\xi)}\left|\frac{\langle\eta,-\xi\rangle^{-1-p}\rho_K^{n-1-p}(\eta) - \lambda^{n-1-p}\langle\eta,-\xi\rangle^{-n}}{n-1-p}\right|\,d\eta \quad (10)\\
&\quad + \int_{B_{K_\delta}(\xi)\setminus B_K(\xi)}\left|\frac{\langle\eta,-\xi\rangle^{-1-p}\rho_{K_\delta}^{n-1-p}(\eta) - \lambda^{n-1-p}\langle\eta,-\xi\rangle^{-n}}{n-1-p}\right|\,d\eta \quad (11)\\
&\quad + \sum_{k=0}^{m-1}\frac{\lambda^{k-p}}{k!|k-p|}\left|A^{(k)}_{K,\xi}(0) - A^{(k)}_{K_\delta,\xi}(0)\right|,
\end{aligned}$$

for $\delta > 0$ small enough. The integrals in expressions (8) and (9) are finite, with

$$\int_0^\lambda\int_0^t t^{-1-p}(t-z)^{m-1}\,dz\,dt = \frac{\lambda^{m-p}}{m(m-p)},$$

since p is a non-integer less than m, and

$$\int_{B_K(\xi)\cap B_{K_\delta}(\xi)}\langle\eta,-\xi\rangle^{-1-p}\,d\eta \le \left(\frac{R}{\lambda}\right)^{1+p}\omega_n.$$

Furthermore, the integrands in expression (10) and (11) are bounded above by

$$\left(\frac{2R}{\lambda}\right)^{1+p}(2R)^{n-1-p} + \lambda^{n-1-p}\left(\frac{2R}{\lambda}\right)^n \qquad \text{if } p < n-1,$$

and

$$\left(\frac{2R}{\lambda}\right)^{1+p}\left(\frac{r}{2}\right)^{n-1-p} + \lambda^{n-1-p}\left(\frac{2R}{\lambda}\right)^n \qquad \text{if } p > n-1,$$

noting that $B_{r/2}(0) \subset K_\delta \subset B_{2R}(0)$ for $\delta < 1/2$.

It is now sufficient to prove

$$\lim_{\delta\to 0}\sup_{\xi\in S^{n-1}}\int_{S^{n-1}}\chi_{B(\xi,\delta)}\,d\eta = 0,$$

where

$$\begin{aligned}
&B(\xi,\delta) = B_K(\xi)\Delta B_{K_\delta}(\xi)\\
&= \left\{\eta\in S^{n-1}\,\middle|\,\rho_K(\eta) \ge \frac{\lambda}{\langle\eta,-\xi\rangle} > \rho_{K_\delta}(\eta) \text{ or } \rho_{K_\delta}(\eta) \ge \frac{\lambda}{\langle\eta,-\xi\rangle} > \rho_K(\eta)\right\}.
\end{aligned}$$

We will prove the equivalent statement

$$\lim_{\delta\to 0}\sup_{\xi\in S^{n-1}}\int_{S^{n-1}}\chi_{B(-\xi,\delta)}\,d\eta=0,$$

where the sign of ξ has changed, so that we may use Figure 1.

Towards this end, fix any $\theta\in S^{n-2}$, and consider Figure 1 specifically when $t=\lambda$. In this case,

$$\left|\overline{OA}\right|=\rho_K(\eta_2)=\lambda\langle\eta_2,\xi\rangle^{-1}\text{ and }\left|\overline{OC}\right|=\rho_{K_\delta}(\eta_1)=\lambda\langle\eta_1,\xi\rangle^{-1}$$

or

$$\left|\overline{OC}\right|=\rho_K(\eta_2)=\lambda\langle\eta_2,\xi\rangle^{-1}\text{ and }\left|\overline{OA}\right|=\rho_{K_\delta}(\eta_1)=\lambda\langle\eta_1,\xi\rangle^{-1}.$$

Any $\eta\in B(-\xi,\delta)$ lying in the right half-plane spanned by ξ and θ will lie between η_1 and η_2. Furthermore, the angle ω converges to zero as δ approaches zero, uniformly with respect to $\xi\in S^{n-1}$ and $\theta\in S^{n-2}$. Indeed, we have

$$0\le\sin\omega\le\frac{2\,\sin\beta\,\sin\gamma}{r\,\sin\alpha}\left|\rho_K(\eta_1)-\rho_{K_\delta}(\eta_1)\right|,$$

using the fact that both K and K_δ contain a ball of radius $r/2$, and with $\sin\alpha$ uniformly bounded away from zero as before. It follows that the spherical measure of $B(-\xi,\delta)$ converges to zero as δ approaches zero, uniformly with respect to $\xi\in S^{n-1}$. □

Lemma 9. *Let $K\subset\mathbb{R}^n$ be a convex body contained in a ball of radius R, and containing a ball of radius r, where both balls are centred at the origin. If*

$$L(n)=8(n-1)\pi^{\frac{n-1}{2}}\left[\Gamma\left(\frac{n+1}{2}\right)\right]^{-1},$$

then

$$|A_{K,\xi}(t)-A_{K,\xi}(s)|\le L(n)\,R^{n-1}\,r^{-1}\,|t-s|$$

for all $s,t\in[-r/2,r/2]$ and $\xi\in S^{n-1}$.

Proof. For $\xi\in S^{n-1}$, Brunn's Theorem implies $f:=A_{K,\xi}^{\frac{1}{n-1}}$ is concave on its support, which includes the interval $[-r,r]$. Let

$$L_0=\max\left\{\left|\frac{f\left(\frac{-3r}{4}\right)-f(-r)}{\frac{-3r}{4}-(-r)}\right|,\left|\frac{f(r)-f\left(\frac{3r}{4}\right)}{r-\frac{3r}{4}}\right|\right\},$$

and suppose $s,t\in[-r/2,r/2]$ are such that $s<t$. If

$$\frac{f(t)-f(s)}{t-s}>0,$$

then

$$\frac{f\left(\frac{-3r}{4}\right)-f(-r)}{\frac{-3r}{4}-(-r)}\ge\frac{f(s)-f\left(\frac{-3r}{4}\right)}{s-\left(\frac{-3r}{4}\right)}\ge\frac{f(t)-f(s)}{t-s}>0\,;$$

otherwise, we will obtain a contradiction of the concavity of f. Similarly, if

$$\frac{f(t)-f(s)}{t-s}<0,$$

then

$$\frac{f(r)-f\left(\frac{3r}{4}\right)}{r-\frac{3r}{4}} \leq \frac{f\left(\frac{3r}{4}\right)-f(t)}{\frac{3r}{4}-t} \leq \frac{f(t)-f(s)}{t-s}<0.$$

Therefore,

$$\left| A_{K,\xi}^{\frac{1}{n-1}}(t) - A_{K,\xi}^{\frac{1}{n-1}}(s) \right| \leq L_0\,|t-s|$$

for all $s,t\in[-r/2,r/2]$. Now, we have

$$|A_{K,\xi}(t)-A_{K,\xi}(s)| \leq (n-1)\left(\max_{t_0\in\mathbb{R}} A_{K,\xi}(t_0)\right)^{\frac{n-2}{n-1}} \left| A_{K,\xi}^{\frac{1}{n-1}}(t) - A_{K,\xi}^{\frac{1}{n-1}}(s) \right|$$

by the Mean Value Theorem, and

$$L_0 \leq \frac{4}{r}\cdot 2\left(\max_{t_0\in\mathbb{R}} A_{K,\xi}(t_0)\right)^{\frac{1}{n-1}} = \frac{8}{r} A_{K,\xi}^{\frac{1}{n-1}}\big(t_K(\xi)\big).$$

Finally, since K is contained in a ball of radius R, we have

$$A_{K,\xi}\big(t_K(\xi)\big) \leq \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} R^{n-1}.$$

Combining these inequalities gives

$$|A_{K,\xi}(t)-A_{K,\xi}(s)| \leq L(n)\,R^{n-1}\,r^{-1}\,|t-s|$$

for all $s,t\in[-r/2,r/2]$ and $\xi\in S^{n-1}$. □

We now prove two lemmas that will be the core of the proof of Theorem 2.

Lemma 10. *Let K be a convex body in $\mathbb{R}^n$ contained in a ball of radius R, and containing a ball of radius r, where both balls are centred at the origin. Let $\{K_\delta\}_{0<\delta<1}$ be as in Lemma 8. If there exists $0<\varepsilon<\frac{r^2}{16}$ so that*

$$\rho(CK, IK) \leq \varepsilon,$$

then, for $\delta>0$ small enough,

$$\int_{S^1} \left|A'_{K_\delta,\xi}(0)\right| d\xi \leq \left(6\pi + \frac{32\,\pi}{\sqrt{3}\,r}\right)\sqrt{\varepsilon} \qquad \text{when } n=2,$$

$$\int_{S^{n-1}} \left|A'_{K_\delta,\xi}(0)\right|^2 d\xi \leq C(n)\left(\sqrt{\varepsilon} + \frac{R^{2n-4}}{r} + \frac{R^{3n-3}}{r^{n+2}}\right)\sqrt{\varepsilon} \qquad \text{when } n\geq 3.$$

Here, $C(n)>0$ are constants depending only on the dimension.

Proof. By Lemma 8, we may choose $0 < \alpha < 1/2$ small enough so that for every $0 < \delta < \alpha$,

$$\sup_{\xi \in S^{n-1}} \sup_{|t| \le r/4} \left| A_{K,\xi}(t) - A_{K_\delta,\xi}(t) \right| \le \varepsilon.$$

We first show that for each $0 < \delta < \alpha$ and $\xi \in S^{n-1}$, there exists a number $c_\delta(\xi)$ with $|c_\delta(\xi)| \le \sqrt{\varepsilon}$ for which

$$\left| A'_{K_\delta,\xi}\big(c_\delta(\xi)\big) \right| \le 3\sqrt{\varepsilon}.$$

Indeed, if $\xi \in S^{n-1}$ is such that $|t_{K_\delta}(\xi)| \le \sqrt{\varepsilon}$, then

$$A'_{K_\delta,\xi}\big(t_{K_\delta}(\xi)\big) = 0,$$

and we may take $c_\delta(\xi) = t_{K_\delta}(\xi)$.

Assume $\xi \in S^{n-1}$ is such that $|t_{K_\delta}(\xi)| > \sqrt{\varepsilon}$. Letting s denote the sign of $t_{K_\delta}(\xi)$, we have

$$\begin{aligned}
&\left| A_{K_\delta,\xi}(s\sqrt{\varepsilon}) - A_{K_\delta,\xi}(0) \right| = A_{K_\delta,\xi}(s\sqrt{\varepsilon}) - A_{K_\delta,\xi}(0) \\
&= \Big(A_{K,\xi}(s\sqrt{\varepsilon}) - A_{K,\xi}(0) \Big) + \Big(A_{K_\delta,\xi}(s\sqrt{\varepsilon}) - A_{K,\xi}(s\sqrt{\varepsilon}) \Big) \\
&\qquad\qquad\qquad\qquad + \Big(A_{K,\xi}(0) - A_{K_\delta,\xi}(0) \Big) \\
&\le \sup_{\xi \in S^{n-1}} \left| \max_{t \in \mathbb{R}} A_{K,\xi}(t) - A_{K,\xi}(0) \right| + 2 \sup_{\xi \in S^{n-1}} \sup_{|t| \le r/4} \left| A_{K,\xi}(t) - A_{K_\delta,\xi}(t) \right| \\
&\le 3\varepsilon.
\end{aligned}$$

It then follows from the Mean Value Theorem that there is a number $c_\delta(\xi)$ with $|c_\delta(\xi)| \le \sqrt{\varepsilon}$ for which

$$\left| A'_{K_\delta,\xi}\big(c_\delta(\xi)\big) \right| = \left| \frac{A_{K_\delta,\xi}(s\sqrt{\varepsilon}) - A_{K_\delta,\xi}(0)}{\sqrt{\varepsilon} - 0} \right| \le 3\sqrt{\varepsilon}.$$

With the numbers $c_\delta(\xi)$ as above, for the case $n = 2$ we have

$$\begin{aligned}
&\int_{S^1} \left| A'_{K_\delta,\xi}(0) \right| d\xi \\
&\le \int_{S^1} \left(\left| A'_{K_\delta,\xi}\big(c_\delta(\xi)\big) \right| + \left| \int_{c_\delta(\xi)}^0 A''_{K_\delta,\xi}(t)\, dt \right| \right) d\xi \\
&\le 6\pi\sqrt{\varepsilon} + \int_{S^1} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \left| A''_{K_\delta,\xi}(t) \right| dt\, d\xi.
\end{aligned} \tag{12}$$

When $0 < \delta < 1/2$, K_δ is contained in a ball of radius $2R$, and contains a ball of radius $r/2$. Lemma 9 then implies

$$\sup_{\xi \in S^{n-1}} \sup_{t \in (-\sqrt{\varepsilon}, \sqrt{\varepsilon})} \left| A'_{K_\delta,\xi}(t) \right| \le \frac{2L(n)\,(2R)^{n-1}}{r}.$$

So, when $n \geq 3$,

$$\begin{aligned}
&\int_{S^{n-1}} \left|A'_{K_\delta,\xi}(0)\right|^2 d\xi \\
&\leq \int_{S^{n-1}} \left(\left|A'_{K_\delta,\xi}\big(c_\delta(\xi)\big)\right|^2 + \left|\int_{c_\delta(\xi)}^0 2A''_{K_\delta,\xi}(t)A'_{K_\delta,\xi}(t)\,dt\right| \right) d\xi \\
&\leq 9\,\omega_n\,\varepsilon + \frac{4L(n)\,(2R)^{n-1}}{r} \int_{S^{n-1}} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \left|A''_{K_\delta,\xi}(t)\right| dt\,d\xi \qquad (13)
\end{aligned}$$

Considering inequalities (12) and (13), we still need to bound

$$\int_{S^{n-1}} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \left|A''_{K_\delta,\xi}(t)\right| dt\,d\xi$$

for arbitrary n. Rearranging the equation

$$\begin{aligned}
\frac{d^2}{dt^2} A^{\frac{1}{n-1}}_{K_\delta,\xi}(t) &= \frac{d}{dt}\left(\frac{1}{n-1} A^{\frac{2-n}{n-1}}_{K_\delta,\xi}(t)\, A'_{K_\delta,\xi}(t)\right) \\
&= \frac{2-n}{(n-1)^2} A^{\frac{3-2n}{n-1}}_{K_\delta,\xi}(t)\, \left(A'_{K_\delta,\xi}(t)\right)^2 + \frac{1}{n-1} A^{\frac{2-n}{n-1}}_{K_\delta,\xi}(t)\, A''_{K_\delta,\xi}(t)
\end{aligned}$$

gives

$$A''_{K_\delta,\xi}(t) = (n-1) A^{\frac{n-2}{n-1}}_{K_\delta,\xi}(t) \frac{d^2}{dt^2} A^{\frac{1}{n-1}}_{K_\delta,\xi}(t) + \frac{n-2}{n-1} \frac{\left(A'_{K_\delta,\xi}(t)\right)^2}{A_{K_\delta,\xi}(t)}.$$

Brunn's Theorem implies that the second derivative of $A^{\frac{1}{n-1}}_{K_\delta,\xi}$ is non-positive for $|t| < r$, so

$$\begin{aligned}
\left|A''_{K_\delta,\xi}(t)\right| &\leq (1-n) A^{\frac{n-2}{n-1}}_{K_\delta,\xi}(t) \frac{d^2}{dt^2} A^{\frac{1}{n-1}}_{K_\delta,\xi}(t) + \frac{n-2}{n-1} \frac{\left(A'_{K_\delta,\xi}(t)\right)^2}{A_{K_\delta,\xi}(t)} \\
&= -A''_{K_\delta,\xi}(t) + 2\left(\frac{n-2}{n-1}\right) \frac{\left(A'_{K_\delta,\xi}(t)\right)^2}{A_{K_\delta,\xi}(t)}.
\end{aligned}$$

Because K_δ contains a ball of radius $r/2$ centred at the origin, we have

$$A_{K_\delta,\xi}(t) \geq \frac{1}{\Gamma\left(\frac{n+1}{2}\right)} \left(\frac{3\pi r^2}{16}\right)^{\frac{n-1}{2}}$$

for $|t| \leq r/4$, and so

$$\begin{aligned}
\frac{n-2}{n-1} \frac{\left(A'_{K_\delta,\xi}(t)\right)^2}{A_{K_\delta,\xi}(t)} &\leq \frac{n-2}{n-1} \Gamma\left(\frac{n+1}{2}\right) \left(\frac{2L(n)\,(2R)^{n-1}}{r}\right)^2 \left(\frac{16}{3\pi r^2}\right)^{\frac{n-1}{2}} \\
&= \frac{\widetilde{L}(n)\, R^{2n-2}}{r^{n+1}}
\end{aligned}$$

for all $|t| \le \sqrt{\varepsilon}$, where $\widetilde{L}(n)$ is a constant depending only on n. Therefore,

$$\begin{aligned}&\int_{S^{n-1}}\int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}}\left|A''_{K_\delta,\xi}(t)\right|dt\,d\xi\\&\le\int_{S^{n-1}}\int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}}\Big(-A''_{K_\delta,\xi}(t)\Big)\,dt\,d\xi+\frac{4\,\omega_n\,\widetilde{L}(n)\,R^{2n-2}}{r^{n+1}}\sqrt{\varepsilon}.\end{aligned}\tag{14}$$

We will bound the first term on the final line above using formula (1). Letting

$$\widetilde{C}(n)=\frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)},$$

formula (1) becomes

$$\begin{aligned}f_{K_\delta}(t)&=\widetilde{C}(n)\int_{S^{n-1}}\int_{|t|}^{\rho_{K_\delta}(\xi)}\frac{1}{r}\left(1-\frac{t^2}{r^2}\right)^{\frac{n-3}{2}}r^{n-1}\,dr\,d\xi\\&=\widetilde{C}(n)\int_{S^{n-1}}\int_{|t|}^{\rho_{K_\delta}(\xi)}r\left(r^2-t^2\right)^{\frac{n-3}{2}}\,dr\,d\xi\\&=\frac{\widetilde{C}(n)}{(n-1)}\int_{S^{n-1}}\left(\rho^2_{K_\delta}(\xi)-t^2\right)^{\frac{n-1}{2}}\,d\xi.\end{aligned}$$

The derivatives of $A_{K_\delta,\xi}$ and $\left(\rho^2_{K_\delta}(\xi)-t^2\right)^{\frac{n-1}{2}}$ are bounded on $(-\sqrt{\varepsilon},\sqrt{\varepsilon})$ uniformly with respect to $\xi\in S^{n-1}$, so

$$f'_{K_\delta}(t)=\frac{1}{\omega_n}\int_{S^{n-1}}A'_{K_\delta,\xi}(t)\,d\xi=-\widetilde{C}(n)\,t\int_{S^{n-1}}\left(\rho^2_{K_\delta}(\xi)-t^2\right)^{\frac{n-3}{2}}\,d\xi.$$

Observing $\widetilde{C}(2)=\pi^{-1}$, and using that $0<\varepsilon<r^2/16$ and $r/2\le\rho_{K_\delta}\le 2R$ for $\delta<1/2$, we have

$$\begin{aligned}&\left|\int_{S^{n-1}}A'_{K_\delta,\xi}(\pm\sqrt{\varepsilon})\,d\xi\right|\\&=\omega_n\left|f'_{K_\delta}(\pm\sqrt{\varepsilon})\right|=\widetilde{C}(n)\,\omega_n\sqrt{\varepsilon}\int_{S^{n-1}}\left(\rho^2_{K_\delta}(\xi)-\varepsilon\right)^{\frac{n-3}{2}}\,d\xi\\&\le\begin{cases}16\,\pi\left(\sqrt{3}\,r\right)^{-1}\sqrt{\varepsilon} & \text{if } n=2,\\ \widetilde{C}(n)\,\omega_n^2\,(2R)^{n-3}\sqrt{\varepsilon} & \text{if } n\ge 3.\end{cases}\end{aligned}$$

This implies

$$\begin{aligned}\left|\int_{S^{n-1}}\int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}}-A''_{K_\delta,\xi}(t)\,dt\,d\xi\right|&=\left|\int_{S^{n-1}}\left(A'_{K_\delta,\xi}(-\sqrt{\varepsilon})-A'_{K_\delta,\xi}(\sqrt{\varepsilon})\right)d\xi\right|\\&\le\begin{cases}32\,\pi\left(\sqrt{3}\,r\right)^{-1}\sqrt{\varepsilon} & \text{if } n=2,\\ 2\,\widetilde{C}(n)\,\omega_n^2\,(2R)^{n-3}\sqrt{\varepsilon} & \text{if } n\ge 3.\end{cases}\end{aligned}\tag{15}$$

Noting that $\widetilde{L}(2) = 0$, inequalities (12), (14), and (15) give

$$\int_{S^1} \left|A'_{K_\delta,\xi}(0)\right| d\xi \le \left(6\pi + \frac{32\pi}{\sqrt{3}\,r}\right)\sqrt{\varepsilon}$$

when $n = 2$. For $n \ge 3$, inequalities (13), (14), and (15) give

$$\int_{S^{n-1}} \left|A'_{K_\delta,\xi}(0)\right|^2 d\xi \le C(n)\left(\sqrt{\varepsilon} + \frac{R^{2n-4}}{r} + \frac{R^{3n-3}}{r^{n+2}}\right)\sqrt{\varepsilon},$$

where $C(n)$ is a constant depending on n. □

Lemma 11. *Let K and L be infinitely smooth convex bodies in $\mathbb{R}^n$ which are contained in a ball of radius R, and contain a ball of radius r, where both balls are centred at the origin. Let $p \in (0, n)$. If $\varepsilon > 0$ is such that*

$$\left\| I_p\left(\|\xi\|_K^{-n+p} - \|\xi\|_L^{-n+p}\right)\right\|_2 \le \varepsilon,$$

then when $n \le 2p$,

$$\rho(K, L) \le C(n, p)\, R^2 r^{\frac{-3n-1+2p}{n+1}} \varepsilon^{\frac{2}{n+1}},$$

and when $n > 2p$,

$$\rho(K, L) \le C(n, p)\, R^2 r^{\frac{-3n-1+2p}{n+1}} \left(\varepsilon^2 + \frac{R^{2(n+1-p)}}{r^2}\right)^{\frac{n-2p}{(n+2-2p)(n+1)}} \varepsilon^{\frac{4}{(n+2-2p)(n+1)}}.$$

Here, $\|\cdot\|_2$ denotes the norm on $L^2(S^{n-1})$, and $C(n, p) > 0$ are constants depending on the dimension and p.

Proof. Define the function

$$f(\xi) := \|\xi\|_K^{-n+p} - \|\xi\|_L^{-n+p}$$

on S^{n-1}. Towards bounding the radial distance between K and L by $\|f\|_2$, the $L^2(S^{n-1})$ norm of f, note that the identity

$$\rho_K(\xi) - \rho_L(\xi) = \rho_K(\xi)\rho_L(\xi)\big(\|\xi\|_L - \|\xi\|_K\big)$$

implies

$$\big|\rho_K(\xi) - \rho_L(\xi)\big| \le R^2\big|\|\xi\|_K - \|\xi\|_L\big|.$$

By Theorem 6, we have

$$\delta_\infty(K^\circ, L^\circ) \le C(n) D^{\frac{n-1}{n+1}}\big(\delta_2(K^\circ, L^\circ)\big)^{\frac{2}{n+1}},$$

where $C(n) > 0$ is a constant depending on n, and D is the diameter of $K^\circ \cup L^\circ$. Both K° and L° are contained in a ball of radius r^{-1} centred at the origin. We then have $D \le 2r^{-1}$, and

$$\sup_{\xi \in S^{n-1}} \big|\|\xi\|_K - \|\xi\|_L\big| \le C(n) r^{\frac{1-n}{n+1}} \left(\int_{S^{n-1}} \big(\|\xi\|_K - \|\xi\|_L\big)^2 d\xi\right)^{\frac{1}{n+1}}$$

for some new constant $C(n)$. There exists a function $g : S^{n-1} \to \mathbb{R}$ such that

$$\big(\|\xi\|_K - \|\xi\|_L\big) g(\xi) = \|\xi\|_K^{-n+p} - \|\xi\|_L^{-n+p}.$$

If $\xi \in S^{n-1}$ is such that $\|\xi\|_K \neq \|\xi\|_L$, then an application of the Mean Value Theorem to the function t^{-n+p} on the interval bounded by $\|\xi\|_K$ and $\|\xi\|_L$ gives

$$|g(\xi)| \geq (n-p)\left(\max\left\{\|\xi\|_K, \|\xi\|_L\right\}\right)^{-n-1+p} \geq (n-p) r^{n+1-p}.$$

Therefore,

$$\big|\|\xi\|_K - \|\xi\|_L\big| \leq (n-p)^{-1} r^{-n-1+p} |f(\xi)|.$$

Combining the above inequalities, we get

$$\sup_{\xi \in S^{n-1}} \big|\rho_K(\xi) - \rho_L(\xi)\big| \leq C(n,p) R^2 r^{\frac{-3n-1+2p}{n+1}} \|f\|_2^{\frac{2}{n+1}}, \tag{16}$$

for some constant $C(n,p)$.

We now compare the L^2 norm of f to that of $I_p(f)$ by considering two separate cases based on the dimension n, as in the proof of Theorem 3.6 in [6]. In both cases, we let $\sum_{m=0}^{\infty} Q_m$ be the condensed harmonic expansion for f, and let $\lambda_m(n,p)$ be the eigenvalues from Lemma 7. As in [6], the condensed harmonic expansion for $I_p f$ is then given by $\sum_{m=0}^{\infty} \lambda_m(n,p) Q_m$.

Assume $n \leq 2p$. An application of Stirling's formula to the equations given in Lemma 7 shows that $\lambda_m(n,p)$ diverges to infinity as m approaches infinity. The eigenvalues are also non-zero, so there is a constant $C(n,p)$ such that $C(n,p)|\lambda_m(n,p)|^2$ is greater than one for all m. Therefore,

$$\begin{aligned}\|f\|_2^2 &= \sum_{m=0}^{\infty} \|Q_m\|_2^2 \\ &\leq C(n,p) \sum_{m=0}^{\infty} \big|\lambda_m(n,p)\big|^2 \|Q_m\|_2^2 = C(n,p) \|I_p(f)\|_2^2 \leq C(n,p)\varepsilon^2.\end{aligned}$$

Combining this inequality with (16) gives the first estimate in the theorem.

Assume $n > 2p$. Hölder's inequality gives

$$\begin{aligned}\|f\|_2^2 &= \sum_{m=0}^{\infty} \|Q_m\|_2^2 \\ &= \sum_{m=0}^{\infty} \left(\big|\lambda_m(n,p)\big|^{\frac{4}{n+2-2p}} \|Q_m\|_2^{\frac{4}{n+2-2p}} \right) \cdot \left(\big|\lambda_m(n,p)\big|^{\frac{-4}{n+2-2p}} \|Q_m\|_2^{\frac{2n-4p}{n+2-2p}} \right) \\ &\leq \left(\sum_{m=0}^{\infty} \big|\lambda_m(n,p)\big|^2 \|Q_m\|_2^2 \right)^{\frac{2}{n+2-2p}} \left(\sum_{m=0}^{\infty} \big|\lambda_m(n,p)\big|^{\frac{-4}{n-2p}} \|Q_m\|_2^2 \right)^{\frac{n-2p}{n+2-2p}},\end{aligned}$$

where we again note that the eigenvalues are all non-zero. It follows from Lemma 7 and Stirling's formula that there is a constant $C(n,p)$ such that

$$\left|\lambda_m(n,p)\right|^{\frac{-4}{n-2p}} \leq C(n,p)m^2$$

for all $m \geq 1$, and

$$\left|\lambda_0(n,p)\right|^{\frac{-4}{n-2p}} \leq C(n,p).$$

Using the identity

$$\|\nabla_o f\|_2^2 = \sum_{m=1}^{\infty} m(m+n-2)\|Q_m\|_2^2 \tag{17}$$

given by Corollary 3.2.12 in [7], we then have

$$\|f\|_2^2 \leq C(n,p)\left(\|I_p(f)\|_2^2\right)^{\frac{2}{n+2-2p}}\left(\|Q_0\|_2^2 + \|\nabla_o f\|_2^2\right)^{\frac{n-2p}{n+2-2p}}.$$

The Minkowski functional of a convex body is the support function of the corresponding polar body, so

$$\nabla_o \|\xi\|_K^{-n+p} = (-n+p)\|\xi\|_K^{-n-1+p}\,\nabla_o h_{K^\circ}(\xi).$$

Because K° is contained in a ball of radius r^{-1}, it follows from Lemma 2.2.1 in [7] that

$$|\nabla_o h_{K^\circ}(\xi)| \leq 2r^{-1}$$

for all $\xi \in S^{n-1}$. We now have

$$\left\|\nabla_o \|\xi\|_K^{-n+p}\right\|_2^2 \leq 4(n-p)^2 R^{2(n+1-p)} r^{-2}\omega_n.$$

This constant bounds the squared L^2 norm of $\nabla_o\|\xi\|_L^{-n+p}$ as well, so

$$\left\|\nabla_o f\right\|_2^2 \leq 16(n-p)^2 R^{2(n+1-p)} r^{-2}\omega_n.$$

Therefore,

$$\|f\|_2^2 \leq C(n,p)\varepsilon^{\frac{4}{n+2-2p}}\left(\varepsilon^2 + R^{2(n+1-p)}r^{-2}\right)^{\frac{n-2p}{n+2-2p}},$$

where the constant $C(n,p) > 0$ is different from before. This inequality with (16) gives the second estimate in the theorem. □

4. Proofs of Stability Results

We are now ready to prove our stability results.

Proof of Theorem 2. Let $\{K_\delta\}_{0<\delta<1}$ be the family of smooth convex bodies from Lemma 8. We will show that $\rho(K_\delta, -K_\delta)$ is small for $0 < \delta < \alpha$, where α is the constant from the proof of Lemma 10. The bounds in the theorem will then follow from

$$\rho(K,-K) \leq \lim_{\delta\to 0}\left(2\rho(K,K_\delta) + \rho(K_\delta,-K_\delta)\right) = \lim_{\delta\to 0}\rho(K_\delta,-K_\delta).$$

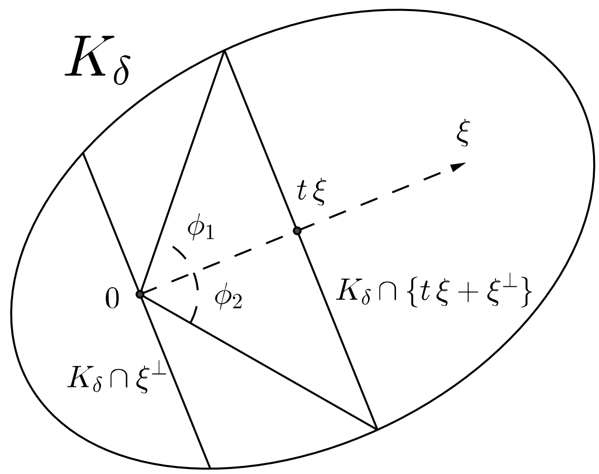

FIGURE 2. K_δ is a convex body in $\mathbb{R}^2$, and $\xi \in S^1$.

We begin by separately considering the case $n = 2$. Let the radial function ρ_{K_δ} be a function of the angle measured counter-clockwise from the positive horizontal axis. For any $\xi \in S^1$, let the angles ϕ_1 and ϕ_2 be functions of $t \in (-r, r)$ as indicated in Figure 4. If ξ corresponds to the angle θ, then the parallel section function for K_δ may be written as

$$A_{K_\delta,\theta}(t) = \rho_{K_\delta}(\theta + \phi_1)\,\sin\phi_1 + \rho_{K_\delta}(\theta - \phi_2)\,\sin\phi_2.$$

Implicit differentiation of

$$\cos\phi_j = \frac{t}{\rho_{K_\delta}\left(\theta - (-1)^j\phi_j\right)} \quad (j = 1, 2)$$

gives

$$\left.\frac{d\phi_j}{dt}\right|_{t=0} = \frac{(-1)}{\rho_{K_\delta}\left(\theta - (-1)^j\frac{\pi}{2}\right)},$$

so

$$A'_{K_\delta,\theta}(0) = -\frac{\rho'_{K_\delta}\left(\theta + \frac{\pi}{2}\right)}{\rho_{K_\delta}\left(\theta + \frac{\pi}{2}\right)} + \frac{\rho'_{K_\delta}\left(\theta - \frac{\pi}{2}\right)}{\rho_{K_\delta}\left(\theta - \frac{\pi}{2}\right)}.$$

Since $f(\phi) := \rho_{K_\delta}(\phi + \pi/2) - \rho_{K_\delta}(\phi - \pi/2)$ is a continuous function on $[0, \pi]$ with

$$f(0) = \rho_{K_\delta}(\pi/2) - \rho_{K_\delta}(-\pi/2) = -\big(\rho_{K_\delta}(-\pi/2) - \rho_{K_\delta}(\pi/2)\big) = -f(\pi),$$

there exists an angle $\theta_0 \in [0, \pi]$ such that $\rho_{K_\delta}(\theta_0 + \pi/2) = \rho_{K_\delta}(\theta_0 - \pi/2)$. With this θ_0, we get the inequality

$$\left|\int_{\theta_0}^{\theta}\left(-\frac{\rho'_{K_\delta}\left(\phi + \frac{\pi}{2}\right)}{\rho_{K_\delta}\left(\phi + \frac{\pi}{2}\right)} + \frac{\rho'_{K_\delta}\left(\phi - \frac{\pi}{2}\right)}{\rho_{K_\delta}\left(\phi - \frac{\pi}{2}\right)}\right)\, d\phi\right| \le \int_0^{2\pi}\left|A'_{K_\delta,\phi}(0)\right|\, d\phi.$$

Integrating the left side of this inequality, and applying Lemma 10 to the right side, gives

$$\left|\log\left(\frac{\rho_{K_\delta}\left(\theta-\frac{\pi}{2}\right)}{\rho_{K_\delta}\left(\theta+\frac{\pi}{2}\right)}\right)\right| \le \left(6\pi+\frac{32\pi}{\sqrt{3}\,r}\right)\sqrt{\varepsilon}.$$

This implies

$$\begin{aligned}1-\exp\left[\left(6\pi+\frac{32\pi}{\sqrt{3}\,r}\right)\sqrt{\varepsilon}\right] &\le \exp\left[-\left(6\pi+\frac{32\pi}{\sqrt{3}\,r}\right)\sqrt{\varepsilon}\right]-1\\ &\le \frac{\rho_{K_\delta}\left(\theta-\frac{\pi}{2}\right)}{\rho_{K_\delta}\left(\theta+\frac{\pi}{2}\right)}-1\\ &\le \exp\left[\left(6\pi+\frac{32\pi}{\sqrt{3}\,r}\right)\ \sqrt{\varepsilon}\right]-1.\end{aligned}$$

It follows that

$$\begin{aligned}-2\left(\exp\left[\left(6\pi+\frac{32\pi}{\sqrt{3}\,r}\right)\sqrt{\varepsilon}\right]-1\right)R &\le \rho_{K_\delta}\left(\theta-\frac{\pi}{2}\right)-\rho_{K_\delta}\left(\theta+\frac{\pi}{2}\right)\\ &\le 2\left(\exp\left[\left(6\pi+\frac{32\pi}{\sqrt{3}\,r}\right)\sqrt{\varepsilon}\right]-1\right)R,\end{aligned}$$

since K_δ is contained in a ball of radius $2R$. Viewing ρ_{K_δ} again as a function of vectors, we have

$$\sup_{\xi\in S^1}|\rho_{K_\delta}(\xi)-\rho_{K_\delta}(-\xi)| \le 2\left(\exp\left[\left(6\pi+\frac{32\pi}{\sqrt{3}\,r}\right)\sqrt{\varepsilon}\right]-1\right)R.$$

The inequality $e^t-1\le 2t$ is valid when $0<t<1$; therefore, if

$$\varepsilon < \left(\frac{\sqrt{3}\,r}{6\sqrt{3}\pi r+32\pi}\right)^2,$$

then

$$\sup_{\xi\in S^1}|\rho_{K_\delta}(\xi)-\rho_{K_\delta}(-\xi)| \le \left(24\pi+\frac{128\pi}{\sqrt{3}\,r}\right)R\sqrt{\varepsilon}.$$

Consider the case when $n>2$. For K_δ with $p=1$, Equation (2) becomes

$$I_2\left(\|x\|_{K_\delta}^{-n+2}-\|-x\|_{K_\delta}^{-n+2}\right)(\xi) = -2\pi i\,(n-2)A'_{K_\delta,\xi}(0),$$

so

$$\begin{aligned}&\left\|I_2\left(\|x\|_{K_\delta}^{-n+2}-\|x\|_{-K_\delta}^{-n+2}\right)\right\|_2 = 2\pi(n-2)\left(\int_{S^{n-1}}\left|A'_{K_\delta,\xi}(0)\right|^2 d\xi\right)^{\frac{1}{2}}\\ &\le \widetilde{C}(n)\left(\sqrt{\varepsilon}+\frac{R^{2n-4}}{r}+\frac{R^{3n-3}}{r^{n+2}}\right)^{\frac{1}{2}}\varepsilon^{\frac{1}{4}}\end{aligned}$$

by Lemma 10. Finally, by Lemma 11,

$$\rho(K_\delta,-K_\delta)\le C(n)\frac{R^2}{r^{\frac{3n-3}{n+1}}}\left(\sqrt{\varepsilon}+\frac{R^{2n-4}}{r}+\frac{R^{3n-3}}{r^{n+2}}\right)^{\frac{1}{n+1}}\varepsilon^{\frac{1}{2(n+1)}}$$

when $n=3$ or 4, and

$$\rho(K_\delta,-K_\delta)\le C(n)\left[\left(\sqrt{\varepsilon}+\frac{R^{2n-4}}{r}+\frac{R^{3n-3}}{r^{n+2}}\right)\sqrt{\varepsilon}+\frac{R^{2(n-1)}}{r^2}\right]^{\frac{n-4}{(n-2)(n+1)}}$$
$$\cdot\left(\sqrt{\varepsilon}+\frac{R^{2n-4}}{r}+\frac{R^{3n-3}}{r^{n+2}}\right)^{\frac{2}{(n-2)(n+1)}}\frac{R^2\varepsilon^{\frac{1}{(n-2)(n+1)}}}{r^{\frac{3n-3}{n+1}}}$$

when $n\ge 5$, where $C(n)>0$ are constants depending on the dimension. □

We now present the proof of our second stability result.

Proof of Theorem 5. Apply Lemma 8 to K and L; let $\{K_\delta\}_{0<\delta<1}$ and $\{L_\delta\}_{0<\delta<1}$ be the resulting families of smooth convex bodies. For each δ, define the constant

$$\varepsilon_\delta:=\sup_{\xi\in S^{n-1}}\left|A^{(p)}_{K_\delta,\xi}(0)-A^{(p)}_{K,\xi}(0)\right|+\sup_{\xi\in S^{n-1}}\left|A^{(p)}_{L_\delta,\xi}(0)-A^{(p)}_{L,\xi}(0)\right|+\varepsilon.$$

Defining the auxiliary function

$$f_\delta(\xi):=\|\xi\|_{K_\delta}^{-n+1+p}-\|\xi\|_{L_\delta}^{-n+1+p},$$

we have

$$\cos\left(\frac{p\pi}{2}\right)I_{1+p}\big(f_\delta(x)+f_\delta(-x)\big)(\xi)+i\sin\left(\frac{p\pi}{2}\right)I_{1+p}\big(f_\delta(x)-f_\delta(-x)\big)(\xi)$$
$$=2\pi(n-1-p)\Big(A^{(p)}_{K_\delta,\xi}(0)-A^{(p)}_{L_\delta,\xi}(0)\Big)$$

from Equation (2). The function of ξ on the left side of this equality is split into its even and odd parts, because I_{1+p} preserves even and odd symmetry. Therefore,

$$\frac{\cos\left(\frac{p\pi}{2}\right)}{\pi(n-1-p)}I_{1+p}\big(f_\delta(x)+f_\delta(-x)\big)(\xi)$$
$$=\Big(A^{(p)}_{K_\delta,\xi}(0)-A^{(p)}_{L_\delta,\xi}(0)\Big)+\Big(A^{(p)}_{K_\delta,-\xi}(0)-A^{(p)}_{L_\delta,-\xi}(0)\Big)$$

and

$$\frac{i\sin\left(\frac{p\pi}{2}\right)}{\pi(n-1-p)}I_{1+p}\big(f_\delta(x)-f_\delta(-x)\big)(\xi)$$
$$=\Big(A^{(p)}_{K_\delta,\xi}(0)-A^{(p)}_{L_\delta,\xi}(0)\Big)-\Big(A^{(p)}_{K_\delta,-\xi}(0)-A^{(p)}_{L_\delta,-\xi}(0)\Big)$$

By the definition of ε_δ,

$$\begin{aligned}\Big|I_{1+p}\big(2f_\delta\big)(\xi)\Big| &\le \Big|I_{1+p}\Big(f_\delta(x)+f_\delta(-x)\Big)(\xi)\Big| + \Big|I_{1+p}\Big(f_\delta(x)-f_\delta(-x)\Big)(\xi)\Big| \\ &\le \frac{2\pi(n-1-p)}{\cos\left({}^{p\pi}/_2\right)}\,\varepsilon_\delta + \frac{2\pi(n-1-p)}{\sin\left({}^{p\pi}/_2\right)}\,\varepsilon_\delta,\end{aligned}$$

which implies

$$\left\|I_{1+p}(f_\delta)\right\|_2 \le \pi\sqrt{\omega_n}\,(n-1-p)\Big(\left|\sec\left({}^{p\pi}/_2\right)\right| + \left|\csc\left({}^{p\pi}/_2\right)\right|\Big)\varepsilon_\delta.$$

Both K_δ and L_δ are contained in a ball of radius $2R$ when $0<\delta<1/2$, and contain a ball of radius $r/2$. It now follows from Lemma 11 that

$$\rho(K_\delta, L_\delta) \le C(n,p)\,R^2 r^{\frac{-3n+1+2p}{n+1}}\,\varepsilon_\delta^{\frac{2}{n+1}}$$

when $n \le 2p+2$, and

$$\rho(K_\delta, L_\delta) \le C(n,p)\,R^2 r^{\frac{-3n+1+2p}{n+1}}\left(\varepsilon_\delta^2 + \frac{R^{2(n-p)}}{r^2}\right)^{\frac{n-2-2p}{(n-2p)(n+1)}}\varepsilon_\delta^{\frac{4}{(n-2p)(n+1)}}$$

when $n > 2p+2$, where $C(n,p)>0$ are constants depending on the dimension and p. Finally, the bounds in the theorem statement follow from the observations

$$\rho(K,L) \le \lim_{\delta\to 0}\Big(\rho(K,K_\delta)+\rho(L,L_\delta)+\rho(K_\delta,L_\delta)\Big) = \lim_{\delta\to 0}\rho(K_\delta,L_\delta),$$

and $\lim_{\delta\to 0}\varepsilon_\delta = \varepsilon$. □

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